Paper IV.—GEOMETRY, MENSURATION, AND THE STEREOMETRICAL TABLEAU.

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No apology, I presume, need be offered for my selection of the subject of which I am about to treat; for, though, at first sight, it may appear to be devoid of interest and practical utility, it is, on the contrary, fraught with paramount importance on account of its relationship to almost all the industries of life.

Geometry is the basis of our public works; to its precepts we are indebted for all the constructive arts, our public edifices, our dwelling-houses, the fortifications of our cities, our ports, canals and roads, and the wondrous architecture of the many floating palaces and towers that thread our streams and rivers or plough the mighty seas. The geometer it is who measures and designs, in their true proportions, the diverse parts of states and territories, and who, by thus bringing together their several points, enables the eye to appreciate the consequences of their relative positions; he it is who directs the use of our engines of war, and to his calculations are made subordinate the movements of armies. Geometry makes the astronomer and guides the navigator, and all sciences are allied to it. It is the foundation of mechanics, hydraulics, and optics; and all the physical sciences are constantly indebted to it.

In a less elevated sphere, geometry teaches us to measure and design or represent our fields, our gardens, and domestic buildings; it enables us to estimate and compare their products and expenditure; it determines inaccessible heights and distances, guides the draughtsman's hand, and presents an infinity of detached usages, often applicable in domestic economy.

The fairer portion of mankind have the keenest, most appreciative perception of its advantages and beauties, as evidenced in the ever-varying combinations so cunningly devised in their designs for patch-work, laces, and embroidery, etc.

Geometry measures extension, comparing portions of space with each other. Its elements are lines, surfaces, and volumes or solids. A portion of space, such as might be filled by a solid body, is itself called in geometry a solid. The boundaries of this geometrical solid are called surfaces, which last may also be conceived as separating space from space as well as bounding it. They constitute a zero of solidity, but have a magnitude of their own, called superficies. It is sometimes suggestive and advantageous, in reasoning about solids and their mode of measurement, to consider them as made up of an infinite number of surfaces or superficies overlying one another like the leaves of a book. If a surface be limited in extent, the boundary on any side is a line, which has neither solidity nor superficial area, having magnitude in length only. Surfaces or superficies may be conceived to be made up of an infinity of lines in juxta-position to each other. If the line is limited in extent, its extremities are points. A point, therefore, is a zero, not only in solidity and superficies, but in length also, having no magnitude or proportion, and retaining only order or position as the sole element of its existence. A line may be conceived as made up of an infinite series of points following each other in close contact or succession. In the position of points, the difference in direction of a first and second point from a third is called an angle. We have here all the elemental conceptions of geometry, viz., a point, a line, a surface, a solid, an angle. From these definitions as data, a vast amount of geometrical science may be deduced by the laws of logic.

The relation of geometry to other sciences is twofold, giving and receiving. To mechanics it gives the only possibility of understanding the laws of motion; and from mechanics it receives the conception of moving points, lines, and surfaces, and thus generating lines, surfaces, and solids. To chemistry it gives the only means of investigating crystallization, polarization, &c., and from chemistry receives new ideas concerning the symmetry of planes. It holds a like relation to botany and zoology in the laws of form and morphology. In the study of the human mind, geometry proposes the question of the foundations of belief, by giving the first examples of demonstration; and from the inquiries thus aroused geometry has received from age to age the new conceptions which have been the base of many new methods of investigation and proof. To theology geometry gives definite conceptions of the order and wisdom of the natural creation, and from theology has been stimulated to many fresh exertions in the investigation of these theological questions.

The history of geometry is divided by Chasles, in his valuable "*Aperçu historique des méthodes en géométrie*," into five periods. The first is that of the Greek geometry, lasting about 1000 years, or till A.D. 550. Then, after a pause of 1000 years, the second period began in the revival of ancient geometry about 1550. A third period was marked in the beginning of the 17th century by Descartes' co-ordinates and the analytical geometry. The fourth period was inaugurated in 1684 by the sublime invention of the differential calculus. The fifth era is marked in our own century by Monge's "Descriptive Geometry," by which he developed the idea of reducing the problems of solid geometry to problems in a plane. One beautiful example of this branch of science may be found in linear perspective, which simply projects the points of a solid upon a plane by straight lines of light from the eye.

Since Chasles' "*Aperçu Historique*" was published, a sixth period has been introduced by the publication of "*Hamilton's Quaternions.*"

Greek geometry, it is said, began with Thales and Pythagoras, who obtained their first ideas from Egypt and from India. Diodorus, Herodotus and Strabo are of opinion that the science of mensuration had its rise among the Egyptians, whom they represent as constrained, on account of the removal or defacing of the land-marks by the annual inundation of the Nile, to devise some method of ascertaining the ancient boundaries, after the waters had subsided. Indeed, the science must have been nearly coeval with the existence of man, for we are told in Holy Writ that Cain built a city; to do which, it is evident, would require some knowledge of a measuring unit, which is the first principle of mensuration. By the same infallible testimony we find that the arts and sciences were cultivated to a considerable extent long before the flood. Jubal was the father of all such as handled the harp and organ, and Tubal-Cain an instructor of every artificer in brass and iron. It is also more than probable that Noah was well acquainted with geometry as practised in his day, for it does not appear that he found any difficulty in building the ark. Be this as it may, it is well known that Egypt was for many ages the mother and nurse of the arts and sciences. From this country they were conveyed into Greece by Thales, about 600 years before the Christian era. Euclid, about 300 years before Christ, established a mathematical school at Alexandria, where Archimedes, Apollonius, Ptolemy, Theon, etc., received instruction from the "prince of geometers." It is, however, reasonable to suppose that before Euclid's time there existed treatises on geometry, for Proclus affirms that Euclid improved many things in the elements of Eudoxus and in those of Theatetus, and established, by the most firm and convincing demonstrations, such propositions as were but superficially explained.

The Pythagorean school demonstrated the incommensurability of the diagonal of a square with its side, and investigated the five regular solids. They had some knowledge of triangles and circles, and were probably acquainted with the fact that the circle and sphere are the largest figures of the same perimeter and surface. About a century after Pythagoras the great Plato and his disciples commenced a course of rapid and astonishing discoveries. The ancient analytic mode of geometrical reasoning consisted in assuming the truth of the theorem to be proved, and then shewing that this implied the truth of those propositions only which were already known to be true. In modern days, the algebraic method, since it allows the introduction of unknown quantities, has taken the name of analytic. Conic sections embrace, as is well known, the study of the curves generated by intersecting a cone by a plane surface; and most marvellous, so to say, are some of them. The circle, the most beautiful of all, we see exemplified in the thousand-and-one forms of every-day life. The ellipse, this almost magic curve, is traced out in the heavens by every planet that revolves about the sun, by every satellite about its primary. It has two centres or foci, the sun or primary in one of them. It approaches in these cases nearly to a circle, or has little eccentricity, while, as in the case of comets, it is lengthened out almost indefinitely. It is produced whenever the intersecting plane cuts the cone in a direction oblique to its axis, the angle formed by this section and the cone's base being at the same time less than that included between the base and side. In imitation of the Divine Architect, man has made it subservient to his requirements; we see it in the arches of all sizes that span our rivers and crown our structures, great and small. The artist applies it as a fitting frame to that most beautiful of ovals, the face of lovely woman. It has very peculiar properties: for instance, the sum of any two lines or radii drawn from its centres to any point in the circumference or perimeter of the figure, is a constant quantity; and a ray of light or sound or heat thrown out or radiated from one of its foci towards the circumference, is reflected to the other focus. You have all heard of the whispering gallery in St. Paul's Cathedral: it is merely an elliptical space surrounded by walls, in which, when two persons stand in the opposite foci, and though comparatively far apart, they can talk to each other in the merest whisper, and without the least danger of being overheard by any other person within the gallery. We have, next, the hyperbola, a peculiar curve, in which not the sum is constant, but the difference of any two radii or lines drawn from the foci to a point in the curve. We obtain it in the cone when cut by a plane at an angle greater than that which the side makes with the base. It is, in a popular sense, a somewhat paradoxical figure, inasmuch as though it be produced ever so far, and in so doing it approach more and more to two certain lines called its asymptotes, and though the lines be close by, yet can it never reach them, even though it should continue onward until the end of time. It has not, in practice, the importance of the other conic sections, but has some uses, such as that of expressing or affording an illustration of the varying pressure of steam while working expansively within a cylinder or other vessel.

Come we now to the parabola, a marvellous curve, indeed. How singular that this figure, which every cone presents when cut by a plane parallel to its side, should be that in which certain comets visit our system—those that enter it but once, and leave it again, never more to return; for it is a curve which, like the hyperbola, but unlike the circle or ellipsis, returns not back upon itself, but of which the opposite sides or branches continually separate more and more, never to meet again. This is, indeed, a most fascinating curve. How very strange that it should happen to be the self-same figure which a stone describes when, thrown obliquely into space, it falls again! An acquaintance with the theory of this most useful arc guides the gunner in throwing his balls and shells into the fortress of the enemy; for, see you, such a curve as they describe when ascending into space, the self-same curve they make again while falling to their destination—the whole arc or curve a true parabola, and each portion thereof the exact half or counterpart of its fellow. Yet, I have not done telling you about this most interesting section. Every jet of water or other liquid issuing under pressure from the side of a reservoir or cistern, or from the pipe of a fire-engine, describes this curve; and hence the distance of projection can be in advance calculated and ascertained. But it has still other applications, to wit: in the construction of the speaking-trumpet, where the mouth lies in the focus of the curve (for it will already have been guessed from the description that it has but one), and the rays of sound striking upon the sides of the tube are projected forth together in a pencil or bundle, so to say, parallel to one another, and direct towards the object spoken to. See, again, this curved line in the mirror which collects the parallel rays of the sun, or other source of light or heat, and reflects them one and all from the surface to the centre or focus, wherein a light or fire may be thereby kindled; and again, in the reflector which, in light-houses, gathers the diverging rays proceeding from the focus, and sends them off together on their errand of humanity. But I, too, am wandering, I find; and though the curve return not upon itself, I must not further follow it in its erratic course.

Within 150 years after Plato's time, this study of the conic sections had been pushed by Apollonius and others to a degree which has scarcely been surpassed by any subsequent geometer.

Geometrical loci are lines and surfaces defined by the fact that every point in the line or surface fulfils one and the same condition of position. Thus, the locus of a point equally removed from any two given points, is the perpendicular drawn from the centre of the line joining these two points; the locus of the vertices of all triangles having the same base and equal areas is a line parallel to the base; the locus of the vertices of all triangles having the same base, and the same ratio between their sides, is the circumference of a circle having its centre in the base produced, and such as to cut it in the required ratio. The investigation of such loci has been, from Plato's day to the present, one of the most fruitful of all sources of geometrical knowledge. Just before the time of Apollonius, Euclid introduced into Geometry a device of reasoning which was exceedingly useful in cases where neither synthesis—that is, direct proof—nor the analytic mode is readily applicable, the *reductio ad absurdum*; it consists in assuming the contrary of your proposition to be true, and then shewing that this implies the truth of what is known to be false. Contemporary with Apollonius was Archimedes, who introduced into geometry the fruitful idea of exhaustion. By calculating inscribed and circumscribed polygons about a circle, and increasing the number of sides until the difference between the external and internal polygons became exceedingly small, Archimedes arrived at the first known ratio between the diameter and circumference of a circle, which he found to be as 1 to $3\frac{1}{7}$. Hipparchus, before Christ, and Ptolemy, after Christ, applied mathematics to astronomy. Vieta, the inventor of algebra, applied it to geometry. Kepler introduced the idea of the infinitesimal, thus perfecting the Archimedean exhaustion, and led to the solution of questions of maxima and minima. Meanwhile, Newton's Fluxions and Leibnitz's Differential Calculus had come into use, and many fine discoveries were made in regard to curves in general; and so fruitful the results, that, as is now well known by every one, the time of an eclipse of the sun may be calculated and foretold for years in advance, and that to the very minute—nay, to within a second almost—of its actual occurrence. Even the reappearance of a comet may be predicted to the very day it heaves in sight, and though it has been absent for a century. In former ages, when these phenomena were unpredicted, they burst upon the world in unexpected moments, carrying terror everywhere, and giving rise to the utmost anxiety and consternation, as if the end of all things were at hand.

Although the elements of Euclid are the groundwork of every mathematical education, yet, many valuable rules are deduced from the higher branches of analysis, and which appear to have little or no dependence upon geometry. The differential analysis has admitted us to the knowledge of truths which would astonish mathematicians of former ages; and to Newton we are principally indebted for discoveries which have greatly advanced the art of mensuration. Artificers of all kinds are indebted to geometry and mensuration for the establishment of their various occupations; and the perfection and consequent value of their labours depend entirely on the near approach they make to the standard of geometrical accuracy. All the great and ingenious devices of mankind owe their origin to this sublime science. By its means the architect draws his plan and erects his edifice. When bridges are to be built over large streams, the most exact acquaintance with geometry is required. In the construction of ships, of every kind, geometrical knowledge is requisite; and the sailor well knows the use of a good astronomer on board his vessel to guide it through the trackless ocean. The sublime science of astronomy is built upon geometrical knowledge; and the telescopic observer keeps up, as it were, a conversation with the heavens. In short, all the elegancies of life, and most of its conveniences, owe their existence to the geometrical art. The vegetable world abounds with productions shaped with more than human cunning; the beautiful tracery observable in the petals of some flowers is really astonishing, and the most exact proportion of the parts is always preserved. In the mineral world, a similar truth forces itself upon the imagination; and wherever the eye of man has been allowed to penetrate, the same geometrical harmony is found among all the parts of created matter. And what is the foundation, the groundwork, upon which this science of geometry is built? Why, so to say, on but one or two elementary propositions, which, if the truths implied of them did not exist, the whole science must fail. One of these fundamental theorems is, that in every plane triangle the sum of the angles is constant and equal to two right angles, or to 180 degrees, whereby, when any two of them are known, the third is found with the utmost facility; and thus are we enabled to arrive at the distances which separate us from inaccessible objects, or which divide those objects from each other. It is thus that the surveyor and engineer, by the help of a base-line of measured length, and the angles observed at each extremity thereof, can, by an easy geometrical construction, or by arithmetical calculation, arrive at the exact breadth of a river which it is intended to bridge. Thus, also, does the astronomer, by adopting a broader base, arrive at the diameter of the earth we inhabit, without the necessity of going round it. The earth, in its turn, is made the basis for computing the distance of the moon, the sun, and the planets; and when, as in the case of the fixed stars, this basis fails,—when the astronomer at opposite ends of the earth, or with a base-line of some eight thousand miles in extent, fails to elicit any difference between the sum of the observed angles and two right angles, he bides his time, and, taking one of them on any convenient day, he takes the second that day six months, when, with the earth in its yearly revolution about the sun, he shall have arrived at the opposite end of the earth's orbit, and have thus secured a base-line of nearly 200,000,000 of miles in extent—though, even so, he is sorely tried, and has almost superhuman difficulties to encounter in solving this great problem of the distance of the stars; for,—will it be believed?—so tremendous, so inconceivable is this distance, that the third angle of the triangle—the one opposite to this immense base of 200 millions of miles—this third angle, I say, is but the fraction of a second, or of the $\frac{36}{100}$th part of a degree. And yet, strange contradiction, so very small is this enormous distance of the nearest fixed star, and from which light, though it travels with the inconceivable velocity of 200,000 miles in every second of time, requires three years to reach us,—so small is it, in comparison with this boundless universe, that there are stars ten times—aye, 10,000 times—more remote, and further still, beyond the space-penetrating powers of the most potent telescopes, such as that of Lord Rosse, which is not less than six feet in diameter, and more than 60 feet in length.

And if this property of the triangle did not exist—if the sum of the three angles were not a constant quantity,—then should we probably have remained ignorant, and forever—not, perhaps, of the size of the earth itself, which may be girdled and submitted to direct measurement, but of the distances and sizes of all objects exterior to the earth or out of our immediate reach, and man deprived of one grand and inexhaustible source of enjoyment.

There is another property of triangles on which I must dwell for a moment: it is this, that when similar or equiangular, their homologous or corresponding sides are proportional; and from this property it occurs that in every right-angled triangle the square of the hypothenuse is equal to the sum of the squares upon the other two sides. These two propositions or theorems, together with the more important one already alluded to, are at the foundation, so to say, of all geometrical science; and all the other theorems and problems of geometry depend intimately upon these for their very existence or solution.

Because the sides and altitudes of similar triangles are proportional, it follows that their areas are as the squares of any of their corresponding dimensions. That is, if the base of a triangle be double that of another, so is its altitude also double; and as twice two are four, the area of the second is four-times that of the first; or if the base of the one be three-times that of the other, so will its altitude, and the area 3-times 3, or 9-times that of the first. Hence, while the lineal dimensions increase as the natural numbers 1, 2, 3, etc., the superficies increase as the squares 1, 4, 9, etc., thereof; and this affords a way of dividing any triangular figure or space into portions which shall be equal to each other, or bear to each other any required ratio. And what is true of similar triangles is also true of all other similar figures—that is, of such as are made up of, or capable of being divided into, an equal number of similar and similarly situated triangles. Again, because every rectilineal figure may be divided into as many triangles as the figure has sides, less by two, and as the sum of the angles of each of the constituent triangles is equal to two right angles, therefore does it follow that the sum of the interior angles of any quadrilateral or four-sided figure is 4 right angles; of any pentagon, the sum of the angles is equal to 6 right angles; in an octagon, 12 right angles; and so on. This important property enables the land-surveyor or engineer, after measuring the angles of any tract of land, to test the accuracy of his triangulation, and detect an error, if there be one, since the sum of all the angles, when taken together, and whatever fractions of a degree or minute they may severally contain, must, of necessity, make up an exact number; and that number must be an even number of right angles—2, 4, 6, 8, 10, or 20, as the case may be,—but never 3, 5, 7, 9, or any other odd number.

I have alluded to the circle as being the most beautiful of all figures, but have as yet said nothing of some very useful properties with which it is endowed. For instance, an angle at its centre is measured by the arc which it subtends; this we all know, and that it should be so does not appear at all strange: it seems, on the contrary, that it should not be otherwise, and hence it suffices to define the thing, or to ennunciate the proposition, to have it credited. But it is singular that when the apex of the angle is in or on the circumference, such angle is but one-half—not less, not more—of the corresponding angle at the centre; and out of this arises the very curious and useful property that all angles in the same segment are equal to each other; also, that every angle in a semi-circle is a right angle. Hence the possibility of drawing a tangent to a circle from any point without it; hence, also, can a right angle be most easily and readily laid out by the draughtsman on his board or paper, or by the surveyor in the field. Hence, again, can the very pretty and useful problem be solved of finding a mean proportional or a geometrical mean between any two given lines, a graphic mode of extracting the square root of the product of any two given numbers, or of finding the side of a square equal in area to that of a given rectangle. But, in the same way as a geometric mean may be found between any two lines, so can either of these last be determined when the other is known and the mean between it and its fellow; and in this way is the engineer enabled to find the radius of a railroad curve, for these curves are usually of vast extent, and, unlike the circle on a board or sheet of paper, the centre cannot be seen nor found, nor can the radius be measured; or if you come across a portion of a stump, and have a curiosity to know the size of tree cut from it, draw any chord across it, bisect it, square its half, divide the product by the versed sine or height or breadth of the segment at its centre, and there will come the rest of the diameter. This is not all; for on a scale 10,000 times more vast, or even millions, does the astronomer compute, and almost in the self-same way, from a knowledge of a minute portion of the arc or orbit, those tremendous circles, eccentric though they be, which satellites sweep out around their primaries, and planets around the sun. Yes, and as the engineer, without a centre or a radius, can follow out his curved track among the woods and waters of the earth, so also does the astronomer trace out his mighty circuits through the starry forests of the dark blue heavens. Aye, even is the erratic comet in this way followed up with eye intent upon its ever-varying direction among the stars, and from the minutest portion of its circuit can the elliptic figure be computed which will enable the time of its periodic return to be predicted to a certainty; and the same course of observation will also tell if the path among the planets be not elliptic, but rather parabolic, or that of some strange meteor which is approaching this world's precincts for the first time, and will leave it, never to return—unless, to be sure, the path described should approach very nearly to the elliptic, and, as in political astronomy, the influence or attraction of some great planet so swerve it on its way, so alter its direction, as to bring about the phenomena of a transformation.

I have just now said that the angle at the circumference of a circle is half the angle at the centre on the same arc, and this property can be turned to great account. In surveying the coast of any country, with a view to laying down on charts and maps its shoals and breakers, the hydrographer takes his angles from points immediately over those of which the positions are to be ascertained to three or more points on shore, of which the distances apart are known. Now, were the angles just alluded to adjacent to any one of the measured distances, the solution of the problem would reduce to that of determining the third angle and other two sides of a triangle, of which two angles and a side were known. But the elements are not in contact; they are not adjacent: how, then, can they be brought together for the purposes of calculation? Why, by this very beautiful property of the circle just enunciated—that an angle having its vertex in the circumference is equal to any other angle similarly situated and bearing upon the same arc, whereby, if a circle be described around three of the points concerned in the problem,—and, as we all know, a circle can be so described,—and if the two angles, taken from the hydrographical point under consideration, are made to travel round the circle till the apex of each of them arrive at the opposite extremities of the base, the whole difficulty will have been made to disappear, and the problem be reduced to that of solving a case in plane trigonometry. In this way did a former pupil of mine, Mr. R. Steckel, now of the Department of Public Works at Ottawa, solve, in a most ingenious and simple manner, the problem of the interpolation of a base-line, or of finding the unknown portion, B C, of a straight line, A B C D, to which three angles had been taken from a fifth point, P, of which the position was to be ascertained. This solution is given at page 251 of my treatise on Geometry, etc., published in 1866; and at page 277 of the same work is a most ingenious solution of a rather difficult problem in the division of lands —that of a quadrilateral into equal or proportional areas, with sides also proportional to those of the whole figure.

There are yet a few other properties of the circle which I must notice ere I take leave of the subject. Thus, two tangents drawn to a circle from any point without it, are equal, and the tangent is perpendicular to that radius which is drawn to its point of contact; and from these circumstances, and the fact already enunciated, that the sum of the three angles of any triangle is equal to two right angles, it becomes possible to calculate the diameter of the earth by merely observing the angle of depression of the horizon from the top of a mountain or other elevated situation, the height of which above the earth's surface is known. A circle can always be described capable of containing a given angle on a given base; hence, for instance, if the height of the flag-staff on the citadel be known, and the angle it subtends from the opposite side of the river, together with the distance across, the height of the citadel itself may be computed. The geometrical solution or construction of this problem, as of many others of practical utility, is given at pages 232 to 331 of my treatise. Out of the fact that an angle at the circumference is half of that at the centre, there arises also the condition that to inscribe a four-sided figure in a circle, its opposite angles, taken together, must be equal to two right angles; and because any two lines or chords which cut one another in a circle have the parts of the one proportional to those of the other, the diameter or radius can be found of a circle of which a zone, or portion included between any two parallel chords, forms part. And, again, out of the circumstance that if two lines be drawn from any point without a circle to the opposite or concave side thereof, these lines are reciprocally proportional to their segments situated without the circle, there arises one of the modes of solving that case of plane trigonometry wherein the three sides of a triangle are given to find the angles; and a tangent drawn from the same exterior point to the circle is a geometrical mean, or a mean proportional between either of the aforementioned lines and its exterior part; whence there arises a mode of running a railroad curve, for instance, through any two given points and tangent to the straight or curved portion of another road.

How much more which I have not time to relate?—and yet how strange, that of this the most regular of all figures, and which man has made subservient to so many purposes of practical utility,—this figure, which it is so very easy to trace out, neither has the circumference nor the area yet been found. Archimedes, as I have already said, found an approximate ratio for the diameter to the circumference, that of 1 to $3\frac{1}{7}$; Metius, a ratio of 113 to 355; and other mathematicians that of 1 to 3·141592, etc. In 1590, Ceulen, who lived in the time of Metius, extended the calculation to 36 decimals, which were engraven on his tomb. He arrived at this result by calculating the chords of successive arcs, each of which was the half of the preceding one; the last arc in this case being the side of a polygon of 36,893,488,147,419,103,233—nearly 37 billions of times 419 millions of sides. The mode of calculation was thereafter greatly simplified by Snell, who, with the help of a polygon of only 5,242,880 sides, carried the approximation to 55 places of figures. The computation was during the last century continued by other mathematicians, who successively carried the number of figures to 75, 100, 128, and 140 decimals.

Notwithstanding that Lambert, in 1761, and Legendre, in his elements of geometry, have proved that the ratio of the diameter of a circle to its circumference cannot be expressed in numbers, the desire to satisfy those who still hoped to find this ratio led other mathematicians to continue adding to these figures. In 1846, 200 decimals had been obtained, and

250 the following year. In 1851 the number was extended to 315; then to 350. Shanks carried it to 527, and in 1853 to 607 places of decimals. When it became evident that the arithmetical expression for this ratio was out of the question, many persons continued to hope for some geometrical solution to the far-famed problem: but it is generally recognized at present that this method is impracticable; and it must be admitted that there has resulted but trifling advantage, if any, from the enormous time and trouble devoted to this famous proposition. The French Academy of Sciences, in 1775, and, soon after, the Royal Society of London, with the view of discouraging such futile and fruitless researches, refused to take further notice of any communication relating to the quadrature of the circle, the trisection of an angle, the duplication of the cube, or perpetual motion. An approximation of 600 figures, or even less, is equivalent to perfect and absolute accuracy; for, let it be remarked, it suffices to take in 17 decimals only to avoid an error of the thousandth part of an inch on the six hundred millions of miles which constitute the length of the orbit of the earth around the sun. Ten decimals will afford the circumference of the earth to within an inch, and 13 decimals within the thousandth part of an inch, or less than a hair's breadth; and, at any rate, the figures already found more than suffice for the purpose of determining with absolute accuracy not only the dimensions and distances of the planets, but also of the most distant stars or nebulæ that man can discover with the help of the most powerful and space-penetrating telescopes, or of those which he might discover with optical instruments 10,000 times more powerful than those which he already possesses.

Alluding, again, to the celebrated problem of the trisection of an angle, I must once more take occasion to protest, thus publicly, as I have already done, though without result, at page 330 of my treatise of 1866, not against the pretended and ridiculous solution of this problem by a certain Mr. Thorpe, of Ottawa, after, as he says,—poor man!—devoting

34 years of his life to its discovery, but against the government of Canada—the Patent-office—for sanctioning the pretended solution by the issue of Letters-Patent corroborative of the same, and thus setting the opinions of the government officials before those of the numerous *savants* of Europe and other countries, who have, and had long before that time, declared the geometrical solution of this problem to be impossible—though there are, of course, as with the circle, modes of approximating to the true solution to within limits as narrow as any that can be assigned.

A few more remaks on some properties of certain plane figures, and I shall have done with this portion of my subject.

In parallelograms (I need not remind you what they are; their etymology is suggestive enough of that), Mr. Steckel shewed that each of the complements about the diameter is a mean proportional between the component parallelograms about the same, a property which I fortunately conceived the idea of applying (see page 190 of my treatise) to the solution of a problem of frequent occurrence in the division of lands by a straight line running through a given point. This problem was previously a matter of some difficulty, whether algebraically or geometrically considered (see page 519 to 522 of "Gillespie's Land-surveying," where the formula runs over three lines of type).

The regular hexagon or six-sided polygon is the only figure, exclusive of the square and equilateral triangle, which will fit together without leaving a space between, as does the octagon, for instance—a fact which may be seen exemplified in paper-hangings, in the patterns of oil-cloths, and in marble and mosaic tilings. Now, the very bee knows its geometry so well, that it builds its cells in hexagons. The square or triangle would have fulfilled the condition of leaving no interstice, no loss of space between the cells; but neither would have been so well adapted to the almost circular shape of the insect's body as is the hexagon. My young friends will, of course, suggest that the circular instead of the hexagonal would have been even a better shape for the bee to dwell or move in. Granted; but circles, like octagons and other figures, will not fit each other without loss of space; and there is another, and, no doubt, much more important consideration to the bee: it is, that with the hexagon each component wall or partition answers for two adjoining cells, whereas with the circle or cylinder a whole one would have been required for each tiny animal, and the necessary quantity of wax thereby nearly doubled.

There is, there has long been, a tendency towards generalization in the exact and other sciences, and with more reason now than ever. Two thousand years ago, when Euclid lived, steam and electricity were unknown; pneumatics, optics and chemistry were not practised; photography was not dreamed of. In those days, one could afford to devote years to the sole study of mathematics. We cannot do so now: life is too short, and there are too many things to learn. Imbued it was with this idea that I wrote my treatise of 1866. Nobody, apparently, had dared before me to lay his sacrilegious hands upon the venerable teachings of the prince and patriarch of geometers; neither have I done so; but what I have done (and that I was not far wrong in acting so, will, I think, be admitted) has been to reduce, by more than one-half, the separate and demonstrable propositions of the Greek geometer, while retaining the whole of his conclusions.

The fifth book I have eliminated altogether. I have removed it from the elements, and given all its teachings in my "principles," making axioms of some and corollaries of others of Euclid's propositions; for, I hold that to conceive and admit the truth of an axiom, there takes place within the mind a certain process of reasoning, however short it be. For instance, equal ratios are equal quantities, and quantities which are equal to the same or to equal quantities are equal to each other. It, therefore, follows, as a mere corollary of this axiom, that "Ratios which are equal to the same or to equal ratios are equal to one another;" hence, I do not see the necessity of making this a demonstrable proposition. Again have I made an axiom of proposition F of Playfair's Euclid, and justifiably so, I take it; for quantities which are made up of the same or of equal quantities are equal to one another; and since ratios are quantities—numerical ones—therefore are "ratios which are composed or made up of the same or equal ratios equal to each other." Of Euclid's 2nd and 3rd propositions, book I., I have made postulates. Of his 22nd I have made my 1st, and thence deduced *his* 1st, as a simple consequence thereof. Why, for instance, make a theorem of the enunciation that two lines parallel to a 3rd are parallel to one another, or that two triangles similar to a third are similar to one another?—for, what constitutes this parallelism of the lines, this similarity of the triangles, but equality of distance in the first and equality of angular space in the second?—hence have I made of the former a corollary to my definition of parallel lines, and of the latter a corollary to my definition of similar figures. Of Euclid's 35th and 36th of the 1st book I have made but one proposition; for Euclid himself, who in his 4th and 8th of the same book places his figures the one upon the other to prove their equality, might in the same way have superposed the equal bases of his parallelograms, so as then to consider them as one and the same base, which would have allowed him to make of the second proposition a mere corollary of the first. Again, with Euclid's two next propositions of the same book, his 37th and 38th, and after his own assertion in his axioms that "things which are halves or doubles of the same thing, or of equal things, are equal," why did he not reduce them to mere corollaries of his 33rd and 34th? I have in the second book added a lemma, which shews how important a proposition is the fifth of that book, and how fruitful in results. I cannot but agree with Clairault in saying that if Euclid considered it necessary to demonstrate such a self-evident proposition as that a line joining two points in the circumference lies entirely within the circle, it must be because he had to answer and confute the objections of obstinate sophists who made it a point to refuse their assent to the most evident truths; for as well might it be attempted to be proven that the diagonal of a square lies within and not without the figure, or that the centre of a circle is within it. What difference is there between finding the centre of a circle or of a portion only of its circumference? And again, what difference between circumscribing a circle about a triangle and making one pass through three given points? Why, then, did Euclid or his commentators make of these problems as many different propositions, when they really constitute but one and the same operation? A different solution of Euclid's 33rd of the 3rd book allows of reducing its three several cases to one; and so of his 35th and 36th of the same book. Similar processes have been followed out by me in reducing in number the dmeonstrable propositions of the 4th book; and in the 5th, which, as already stated, I have put among the principles, the substitution for "magnitude" of the word "quantity," with its signification defined, to comprise numerical as well as other quantities, has allowed of my reasoning on numbers, and giving, as I have done, the mode of arriving at the numerical and practical solution of the many problems propounded in my work. In Euclid's sixth, why should 14 and 15 be separate theorems, in view of axiom two, which sets forth that what is true of the whole is true of the half? These citations will suffice to give an idea of the process of reduction and generalization followed out by me in the geometry of lines and surfaces; and in the same way have I modified the ordinary demonstrations of solid geometry, and of plane and spherical trigonometry. Nor is my treatise less strictly logical in all its teachings than that of Euclid, every proposition depending for its demonstration or solution on those that came before, and in no way on those that follow.

MENSURATION.

AREAS.

Every triangle, it is evident, is the half, the exact half, of its corresponding parallelogram. Now, the parallelogram is, in area, equal to the rectangle of the same base and altitude; for, if the oblique or triangular portion be cut from one end and added to the other, the figure becomes a rectangle; and as the area of a rectangle is equal to the product of the number of units in its base and altitude, it follows that the area of any triangle is equal to half the product of its length and breadth. This, then, may be adopted as an element into which all plane figures can be divided, and their component areas made up separately and put together. In the case of the regular polygons, the computation of their areas becomes simplified, as they can be divided from the centre into as many equal triangles as there are sides. The area of the trapezium is half the product of its altitude into the sum of its parallel sides.

A sector of a circle is nothing but a triangle of equal altitude throughout, or having a circular base, every point of which is equidistant from the apex; and its area is, therefore, equal to the half-product of its base and altitude; for its arched base may be conceived to be divided into a number of parts, such that each of them shall be, without sensible error, a straight line, and hence the rule; for it is evidently the same thing to compute separately and take the sum of the component triangles of the sector, or to add together their contiguous bases and multiply, once for all, by the altitude or radius. Again, the whole circle is but made up of contiguous sectors or triangles, whence it follows that the area of any circle is equal to the half-product of its circumference and radius. Next, we have to consider among plane figures the segment of a circle, or that which is included between a chord and its corresponding arc; and this is evidently equal to the area of the sector, less the area of the triangle formed by the chord and radii. Now, the lune, a figure like the new moon, and hence its name, formed of two non-concentric arcs of the same or different radii: the area of this figure is the difference of its two component segments, so that a mere repetition of the process just described will measure its superficies. The zone or portion of a circle between two parallel chords can also be conceived as the difference between two segments, or as made up of a trapezium and two segments, and its area found accordingly. Concentric and eccentric rings are, of course, equal in area to the difference of their component circles. Of the ellipse, the area is equal to the product of its diameters into decimal ·7854; for it is found, in the manner hereinabove set forth, that of a circle whose diameter is one, the area is ·7854: in other words, the area of the circle is about 78½ per cent. of its circumscribing square, so that this area is more quickly arrived at by squaring the diameter and reducing the result in the required ratio; and the ellipse being analogous to the circle, its area is found in a corresponding manner. The area of the parabola, I may add, is just ⅔ of its circumscribing rectangle.

There is a more general mode of arriving at the areas of all plane figures; it is by dividing them into a number of trapeziums by a series of equidistant parallel lines or ordinates, and of multiplying the common breadth or distance between the lines by the sum or combined length of all of them except the first and last, of which one-half only must be taken; and this applies to every imaginable figure, in which, the closer the ordinates, the more accurate, of course, the area. There are some few figures bounded by curvilineal lines, of which it is, nevertheless, easy to compute the areas; for instance, where a convex portion thereof is compensated by a corresponding concavity, as in the developed surfaces of intersecting vaults or arched ceilings, or, on a smaller scale, the developed area of the elbow of a pipe or cylinder of any kind. Finally, there is a mode of measuring the surfaces of very irregularly-outlined figures by a system of compensating lines which are drawn so as to include such portions of the superficial space outside the figure as may make up for, or be equivalent to, the portions left without the lines.

SOLIDS.

The measurement of solids comprises that of their surfaces as well as that of their volumes or solidities. Solids may be classified as prisms and prismoids, cylinders and cylindroids, pyramids, cones and conoids, spindles, spheres and spheroids. Prisms are solids which have two equal and parallel ends or bases, and of which all the other faces are parallelograms; so that the prism is equal in diameter or breadth throughout its whole extent, and different in this respect from the pyramid or prismoid of which the sides incline or taper towards one end of the figure. A prism may have a triangle for its base or any other figure, and is called, after the nature of such base, triangular, quadrangular, pentagonal, and so forth. The cylinder is nothing but a prism having a circular base or polygon of an infinite number of sides. Among prisms, the parallelopipedon is that of which the opposite faces are parallel, as the name implies, as in the cube; in consequence of which, any side or face of this or similar solids may be assumed as the base. We all know what a pyramid is, and so of a cone, which is only a pyramid with a circular base. A conoid is a species of cone rounded off at the top like a pear, or like the ice-cone at the Montmorency Falls. It may be conceived to be generated or traced out in space by the revolution of a parabola, or other like figure, about its axis; and it goes by the name of the generating curve, as parabolic, hyperbolic conoid. The frustum of a pyramid, cone, or conoid, is that portion of the solid which remains after the apex has been removed; and it is said to be contained between parallel bases when the cutting plane is parallel to that on which the solid stands. All these solids may be right or inclined; and if so, they require to be so designated, as a right pentagonal prism, an inclined octagonal pyramid, an oblique cone or conoid. The spindle, as its popular name implies, is a well-known form, being circular in its cross-section, and tapering from its centre towards the ends: it may be generated in space by the revolution of an arc of a circle, or of an ellipse, or by a parabola or hyperbola, around a line which is called the axis of the spindle, and, like the conoid, derives its distinctive name from that of the generating curve, as a circular spindle, an elliptic, parabolic or hyperbolic spindle,—not that the spindle itself, as a whole, is a very important solid, but that its middle frustum is the geometrical representation of almost every form of cask, the world over. The sphere, that most beautiful of all solid forms, and which, as I have already stated, contains within itself more space or volume than any other figure of equal superficies; the spheroid or flattened sphere, the figure of this earth of ours, and of the moon, and sun, and planets, which, one and all, are flattened at the poles and protuberant at the equator; and, finally, the prolate, or elongated spheroid, make up the varied classes or families of solids, or space-enclosing figures, with their frustums, segments, ungulas or hoofs, and other sections which the limits of a lecture will not allow me more fully to define.

THE STEREOMETRICAL TABLEAU.

I come now to the more immediate object of this lecture, that which has been, perhaps, to some extent, instrumental in procuring for me the honor of an invitation by the Literary and Historical Society of Quebec to read a paper within the classic precincts of its historic halls; and may I hope I shall have treated the subject in a way to warrant the courtesy and be of some interest to the very numerous and highly appreciative audience, the *élite*, the aristocracy, so to say, of the educated or well-read portion of the community, which has on this evening honored me with its attendance and kind and flattering attention. I allude, of course, to the

Stereometrical Tableau which you see here before you, and of which it behoves me to say something, even at the risk of appearing partial to myself. This *tableau*, or board, which is made up of some 200 models, each of which can be removed and replaced at pleasure, and put into the hands of the pupil for examination, comprises almost all the elementary forms which it is possible to conceive. Among them, those which I have just now enumerated, as prisms and prismoids, cylinders and cylindroids, both right and oblique, and the frusta and ungulae of these bodies; pyramids, cones and conoids, right and oblique with their frusta and ungulae or hoofs; the sphere, with its subdivisions into hemisphere, quarter, half-quarter or tri-rectangular pyramid, segments, zones, frusta and ungulae, and many other sections of this solid; the prolate and oblate spheroid, with their many sections and subdivisions; spindles and their sections, including models of casks of all varieties; the five regular polyhedrons or so-called platonic bodies, though known before the time of Plato: there are also a host of other varied forms, such as convex and concave cones and other solids, and frusta and ungulae of the same; concentric and eccentric rings, and a certain number of compound figures, made up of, or capable of being subdivided into, the elementary solids just enumerated.

My object in the preparation of this *Tableau* has been to generalize and make easy and popular the study of solid forms, and the mode of measuring their surfaces and their solidities or volumes. The lateral faces or sides and the opposite or parallel bases or ends of these solids afford all the plane figures, from the triangle, square, and polygon, to the circle, ellipse, and parabola, etc., with their component sectors, segments, zones, and lunes. Of these you have already learned to estimate the superficies. The models also afford examples of convex superficies, inclusive of the spherical triangle and polygon, the spherical zone and lune and segment, and of all the component parts thereof. The mode of arriving at the areas of these is to assimilate them to plane figures by a division of the concave or convex surface by equidistant ordinates drawn from the extremity of the fixed axis of the solid, or that around which the generating curve is supposed to have rotated in sweeping out the solid. The distance between the ordinates, each of which, from the description just given, is, of course, a circle or portion of a circle, is made so small as to allow of the intervening arc being considered a straight line or thereabouts. The figures thus traced out upon the solid under consideration are either continuous zones or portions thereof, and are to be considered as trapeziums—the first continuous, the others not so, or only partly so. Now, to get the area of a trapezium, which, as already defined, is a plane figure with two parallel sides,—and these last may, of course, be circular or curved as well as straight,—the half-sum of such sides is multiplied into the perpendicular distance between them, or the height or breadth of the figure; and the same rule applied to the curved area to be computed is stated thus: to the half-sum of the lengths of the end ordinates—that is, of the first and last—add the sum of all the other ordinates or arcs and circles, and multiply the whole by the distance between the ordinates or by the breadth of the component zone. The combined length of the circles or circumferences is easily obtainable from a multiplication of the sum of their diameters by $3\frac{1}{7}$, or, more correctly, 3·1416. This rule, as applied by me (page 669 of my geometry) to a hemisphere of 263 units in diameter, and with only four ordinates or five segments or zones, brings out the result within less than one per cent. of the truth; while with nine ordinates, the result is erroneous to the extent of only the sixth part of one per cent.; and with 19 ordinates, or 20 sections, the $\frac{1}{40}$ of one per cent., or within $\frac{1}{4000}$ of the true content. Not that I insist, however, on this mode of measurement for convex or concave superficies, where there are other rules which, as in the case of the perfect sphere or spheroid, or segments of those bodies, bring out their curved areas exact; but the great and manifest advantage of this general system is, that its accuracy is independent of the shape of the body to be measured; while, if the rule for a sphere, for instance, were applied to a body not strictly spherical, the result might prove erroneous to a far greater extent than if arrived at by the system of equidistant ordinates,—to say nothing of the great advantage to the practical measurer of having to store his memory with but one general rule applicable to all cases, and this rule the same as that for plane figures, whereby the whole range of areas or superficies, whether plane or convex, becomes submitted to one and the same formula, to wit: a subdivision by equidistant parallel lines into figures, every one of which is a trapezium—whether continuous or non-continuous, it matters not. It now remains to compute the volumes or cubical contents of the solids of the *Tableau*, and, as already stated, they comprise all known elementary forms; and here it is that I lay special claim to the introduction of a system of mensuration which is not approximately accurate as applied to the great majority of geometrical forms, but of which the absolute accuracy is proved and undoubted. The rule is simply what it purports to be, as printed at the top of the *tableau*, *i.e.*: "To the sum of the parallel end areas add four-times the middle area, and multiply the whole by $\frac{1}{6}$ part of the height or length of the solid." The word "*parallel*" is introduced as a reminder that the opposite ends or bases must be contained between parallel planes, or, if not so originally, that they must be made so by subdivision or decomposition of the solid into its constituent elements. The whole difficulty is, therefore, reduced, by my system, to measuring the areas of the opposite bases and middle section, the remainder of the work being a mere multiplication; so that the proposed formula renders this branch of study of such easy and general application that the art or science may now be taught in a few lessons where it formerly required months, or even years. Take up, for instance, the segment of a conoid or spheroid cut off by a plane inclined in any way to the axis of the solid, a figure such as would be presented by the space occupied by any liquid or fluid substance in a vessel of this shape when inclined to the horizon; look at the preliminary labour required by the old rules of finding out the axis or diameters of the entire solid of which the segment under consideration forms a part, and these factors are necessary as elements in the computation. My system dispenses with all this, and the solid, whatever it may be, is taken hold of and submitted to direct measurement, without in any way inquiring about the size or form of body of which it is a section. But for another reason is this study excluded from general education, because, with ordinary rules, the higher calculus is often indispensable; and as even when it has been taught and learnt, it is as soon forgotten, therefore can these ordinary rules be of little or no use to the practical measurer, even when supplied with all the necessary books and data for working out his problems.

It may be objected that, for the prism and cylinder, for instance, the ordinary rule is even more simple than the prismoidal one. Of course it is; but it flows of itself directly and immediately from the formula. Take up a prism: I have defined it to be of equal breadth throughout; then is its middle section, or any other, when made parallel to the base or end, equal in area to such base; and the argument occurs that six-times this area into one-sixth the altitude reduces to the more simple enunciation of once the area into the whole altitude. Again, in the case of the pyramid or cone, the half-way diameter or breadth is just one-half of what it is at the base; and as the half of one-half, or the product of $\frac{1}{2}\times\frac{1}{2}$, is $\frac{1}{4}$, therefore is the half-way area a quarter of that at the base. The pupil who has already learnt this, sees it at a glance, or recalls it to his memory, and reasons thus: four-times the middle area is equal to the base; and twice the base (for the upper area here is zero) into $\frac{1}{6}$ the altitude is identical with the ordinary rule of once the base into $\frac{1}{3}$ the altitude, or $\frac{1}{3}$ the product of the base and altitude. There is one more case in which the old or ordinary rule is apparently more simple than the general formula; it is when we have to do with a paraboloid, of which the volume is just one-half of its corresponding cylinder.

But here we have done with the comparative advantages of the old rules, and in all other cases the formula is exceedingly more simple. Take, for instance, the frustum of a pyramid; and, first of all, how know you that it is one, except by measuring its upper and under edges, and comparing their respective lengths to find out—which you must do—that due proportionality exists between them; else is the body not the frustum of a pyramid, and, therefore, not subject to the rule? But, granted even that it is the figure you do take it for, see you the trouble of getting a mean proportional between the areas of its opposite bases, which includes a lengthy multiplication of those areas and a laborious extraction of the square root of the product, which very few persons know how to work out? How much more simple to arrive at the arithmetical mean of the opposite diameters and the middle area therefrom?—and if the figure be the frustum of a cone, the three diameters are squared, the square of the middle one taken four-times, and the whole multiplied together, that is, their sum, by decimal ·7854, and the result by $\frac{1}{6}$ the altitude of the frustum; and as this calculation has to be repeated every day, in all parts of the world, in computing the contents of tubs and vats of all imaginable sorts and sizes, the saving in time and trouble is certainly most worthy of consideration.

But suppose this prism or cylinder, this pyramid or cone, this conoid, or this frustum, to be not truly such a figure; let it differ but ever so slightly from what it should be to enable it to be submitted to ordinary rules,—then, if such rules be made use of in computing its contents, adieu to all accuracy, since the very element by which the body differs from its geometrical prototype—that is, its intermediate diameter—is not taken the least notice of; while the prismoidal formula, on the contrary, takes in this ever-varying element, this half-way breadth between the top and bottom in a tub or vat, between the bung and head, as in a cask, and gives a result, in 99 cases out of 100, more true than any other system where this important and indispensable element of variation is not attended to. With regard to the sphere or spheroid, each of its opposite bases is a zero of superficies, as a plane can touch either of them only in one point, and the sum of the areas in this case is four-times the middle area. And how correct this is you shall directly see, for, by ordinary rules you are taught to multiply the convex area of the sphere by $\frac{1}{3}$ of the radius; but this convex area is precisely equal to four-times the middle section, or to four great circles of the sphere, and $\frac{1}{3}$ the radius is the same thing as $\frac{1}{6}$ the diameter or altitude,—so that here again, you see, as in the case of the prism or cylinder, the pyramid or cone, the proof direct of the accuracy of the rule. Now, take up a hemisphere: the half-way area is easily shewn to be just $\frac{3}{4}$ of that at the base; and as four-times $\frac{3}{4}$ are three, and 3 and 1 are four, four great circles into $\frac{1}{6}$ the altitude of the half-sphere gives, of course, half the solidity just obtained, or that of the hemisphere under consideration. The same is true of the flattened or of the elongated sphere or spheroid and ellipsoid, and of the half thereof, and whether the cutting plane or base be perpendicular or not to either axis of the solid; and the areas which enter as elements into the computation of the cubical contents are always ellipses, and, what is more, they are similar or proportional ellipses; so that, from knowing any one of the diameters of the middle section, the area can be directly found by rule-of-three, since, as already shewn, the areas of similar figures are proportional to the squares of any of their corresponding dimensions. The exactitude of the formula, as applied to any other segment or zone of a sphere, is fully demonstrated at paragraph 1529 of my Mensuration; its very near approach to truth, in the case of spindles or their frusta, at paragraphs 1531 and 1574; and its absolute accuracy in the case of any segment of a spheroid, the right or inclined paraboloid, or hyperboloid, at paragraphs 1560 to 1567 of my work.

Now, there may be some curiosity to know how the idea occurred to me of treating every solid as a prismoid by this one and undeviating formula; it is this: taking up the ordinary prismoid, I find its definition to read thus:—"Any "solid having for its opposite bases parallel rectangles; and, "by extension, any solid having for its parallel bases plane "figures with parallel sides." Now, please observe that the only condition expressed or implied in this definition of a prismoid is the parallelism of the sides, and nothing more. Such parallelism does not exclude the proportionality of the sides; therefore is the frustum of a pyramid, to me, a prismoid; and this is what no one before me, that I am aware of, at least, appears to have conceived; for in no treatise have I ever seen the prismoidal formula applied to the frustum of a pyramid or cone. Look, again, at the rectangular prismoid, and as no ratio of the sides is implied, let the ratio be infinite; or, in other words, let one of the parallel sides approach towards the other until they meet and form a single line, or edge, or arris; and then have we the wedge, which is, therefore, another prismoid. Again, let this edge, or line, or arris, become shorter, and still shorter, until it dwindles to a point; and then have we a pyramid, which is also a prismoid, to all intents and purposes. And since a line or edge may become a mere point, so, conversely, may a point become a line, whereby a prismoid, originally square or rectangular, may have one or both of its opposite bases modified into an almost infinite variety of similar or dissimilar figures, as I have shewn at pages 713 to 718 of my treatise; and the more general enunciation is thereafter arrived at, that a prismoid may have for its parallel bases any two figures, whether equal or unequal, similar or dissimilar; any figure and a line parallel to the plane thereof, as in the wedge; any figure and a point, as in the cone and pyramid; any two lines not parallel, but situated in parallel planes. Other definitions may be given more concise than this, more technical and scientific, as that "the prismoid is swept out in space by the revolution of a straight line around two parallel planes of any form whatever, and irrespective of the relative velocities of the two extremities of the generating line;" but the former gives the best idea of the form of solid, as it defines the figure of its ends or bases.

The *Tableau* you will find upon inspection to offer a variety of forms; for instance, one base a square, the other also a square of greater or less size, but turned diagonally as regards the other, the middle base an octagon. Again have we prismoids or cylindroids of which one base is a circle, the other an ellipse, or two ellipses of equal or different size, the longer diameter of the one corresponding to the shorter diameter of the other, and other forms may be conceived in almost endless variety; and of all, without exception, the formula gives the true cubical contents, each of the models exhibiting at a glance, by means of the pencil-line to be seen upon it, the nature and dimensions of the middle section. A word in relation to the regular polyhedrons which are also among the models on the board. Of these bodies there are but five, strange enough to say; and yet, the conclusion is immediate and inevitable, for it takes at least three planes to make a solid angle; and as the sum of these plane angles must be less than 360°, or four right angles, as otherwise the solid angle would then cease to exist and become a plane surface, it suffices to examine which are the regular polygons, whose angles, taken in threes and fours and fives, etc., make up an angle less than four right angles. The equilateral triangle can be put together in 3's and 4's and 5's, which affords the tetrahedron, octahedron, and icosahedron. It cannot be taken in sixes, as six-times 60° = 360°; and therefore can we have no other regular solid with its faces equilateral triangles than the three just enumerated. The right angle can be taken in 3's only : two would not enclose a space, and four would form a plane ; hence is the perfect cube the only solid that can be formed with square faces. Lastly, the pentagon, of which the angle is 108°, supplies the dodecahedron, or 12-sided figure, the sum of its three plane angles being 324°. The regular hexagon cannot be made to answer, as its angle is 120°, and three of them would make up four right angles, as you see they do when looking at their exquisite arrangement in the beehive, or the not less beautiful symmetry displayed in the nest of the common wasp. *A fortiori*, then, can heptagons or any other polygon not be used to build these solids with ; and hence, again, as just stated, can there be but five, and only five, of these platonic forms. As to their mode of measurement, the cube or hexahedron is a mere prism, the tetrahedron a mere pyramid, the octahedron two pyramids base to base, and the two last, the 12 and 20-sided figures, made up of as many pyramids, having their common apex in the centre of the solid or of its imaginary circumscribing sphere, whereby will the computation of one of these component pyramids afford the volume of the whole.

As to the measurement of compound figures like the frustum of a cone or pyramid, or of any other body, between bases that are not parallel, the solid is resolved or decomposed by a section parallel to the base and passing through the lowermost edge of the frustum's upper base into two portions, one of which is a frustum proper, the other a hoof, or ungula, or pyramid, or some other figure, as the case may be. The common buoy is thus resolved into a cone and segment of a sphere ; a gun or mortar, the frustum of a cone or cylinder, with a hemisphere or segment of a spheroid ; the Turkish or Moorish dome, or pinnacle, or spire, the middle frustum of a sphere or spheroid, surmounted by a hollow or concave cone, and so on. If a spherical cone be proposed, it is evident that it can be conceived and treated under two aspects—first, as a cone proper, with the addition of a segment of that sphere of which the cone forms part; or (and this applies to any spherical pyramid, or frustum of such pyramid or shell, or hollow sphere, or any portion of a shell,) to the sum of its end-areas, spherical though they be, add four-times the parallel and middle area, and their sum into $\frac{1}{6}$ the altitude will be the true content. Of course, I need hardly remark that in dealing with a hollow sphere or shell, the content is more quickly arrived at by applying the formula direct to and taking the difference of the inner and outer or component spheres.

I have said that the formula applies exactly and demonstrably to the great majority of solids. From this it is, of course, inferred that there are some exceptions, as in the case of hoofs, ungulas and spindles, but in the same way as the cask or middle frustum of a spindle is measured to within almost perfect accuracy—say to within the quarter or $\frac{1}{10}$ or $\frac{1}{20}$ of one per cent., or a half-pint on a hogshead (see pages 707, 708, and 709 of my treatise), by working upon its half, or by taking the bung diameter as that of one of its ends or bases, and for the middle area that which is at its quarter, or half-way between the head and bung,—in the same way, I say, as this is done, so in the hoof and ungula of any solid may almost absolute accuracy be attained by a subdivision of the body into parallel slices, two or three of them generally sufficing, or four or five when the minutest accuracy is insisted on, just as we approach nearer and more near to the circumference of any circle of which the diameter is known, by taking in more decimals. And in the same manner may hollow or concave cones or cylinders, or if they be convex or swollen out, or other bodies be decomposed and measured, which are not true geometrical figures, and thus the one, and only *one*, most simple formula maintained, made use of, and applied in every conceivable case, without the necessity of learning or remembering any other. The subdivisions, the decomposing planes of section, may be made equidistant, and the grand result thus arrived at, that one universal rule, one formula, will measure all solids and all surfaces ; for, though I have not yet alluded to the fact, it is demonstrable—moreover, it is shewn—that when plane figures, surfaces, are cut up by equidistant ordinates, their areas, the area of each of them is equal to the sum of its bases and four-times the middle base or section into $\frac{1}{6}$ the altitude of the figure. Now, if any solid be divided by a series of equidistant planes, and any surface by a similar series of equidistant lines into portions of equal altitude, and if intermediary or half-way sections be conceived in either case, we get the universal formula that their contents, solid or superficial, as the case may be, are equal to the sum of the extreme bases, together with twice the sum of all the other end-bases, and four-times the sum of all the half-way sections into $\frac{1}{6}$ the common altitude ; the bases and sections being, of course, superficial or linear, according as the figure is a space-enclosing one, or a mere surface, plane or curved though it be. I have already alluded to the tendency towards generalization in almost everything: in physics and in chemistry are all phenomena submitted, so to say, to some general, some universal law or rule. The mechanical powers may be said to constitute but one, and are, at any rate, similar in this, subjected to this law, that what they give in power they consume in time or space. The one measuring unit, the metre, is forcing its way throughout the world, and in its wake must follow one unit for all surfaces, one for all capacities. So are coin and currency becoming simplified. Complete the scheme, and to the sameness and identity of the unit add one general mode of computation; then will not only the unification of the currency be brought about, as proposed in France, and the project most ably abetted in this Dominion by our fellow-countryman, R. S. M. Bouchette, Esq., but it shall be the more universal scheme, the one grand unification and generalization of all science, one universal language, whereby nations may commune, converse with one another,—some day, one religion, one shepherd, and one flock,—the millennium, indeed. But do you see the immense advantages of this generalization? See you its untold consequences? Look at the time now necessarily devoted to the mere reduction and translation of all these ever-varying rules and units; they fill a thousand volumes, while the time and trouble devoted to their compilation might be so much more profitably employed, now-a-days, in the study of new sciences and new arts conducive to the greater happiness and welfare of mankind.

Now that we have, so to say, glanced at all the figures on the "*Tableau*," with their more or less regularity of outline, their plane and curved surfaces, the question still arises: "How are irregular bodies of all kinds to be measured, such as statuary, bronzes, carving, and the like?" And that this lecture, so far as the limits of a lecture will allow, may be complete in itself, and go over the whole ground foreshadowed in its title, it behoves me, in a few words, to supply the necessary information. It is very simple, and, in fact, more so, to arrive at the exact cubical contents of a very irregular body than of one of more exact form; for, with the latter an attempt is always made at direct measurement, while with the former a mechanical process is followed, which solves the problem in a manner most expeditious and most satisfactory. Take up a statue or other carved figure, or the capital of an Ionic or Corinthian column, a square or cylindrical vessel capable of containing it. Pour water into this vessel until it reach to above the top of the object to be measured, and mark the height at which the water stands; then remove the object, and again note how high the water stands, when the difference will immediately afford the volume sought. If the substance of the body or of the containing vessel be of an absorbent nature, use sand or some like substance instead of water. There are still other modes of arriving at correct conclusions. The specifie gravity of any body is its weight compared with that of water. Suppose, then, that tables have been prepared wherein the ratio of weight of every substance is given to its equivalent of water, or its absolute weight without regard to the equivalent: take up from off the public highway a shapeless stone, and weigh it; compare its weight by rule-of-three with that of a cubic foot, or inch, or yard of the same substance, and hence do you arrive directly at its cubical contents. Conversely, if the weight be required of some object which cannot be submitted to direct computation by putting in a balance, but if its volume can be arrived at, then also can its weight be ascertained by a simple rule of proportion.

Now, let it be required to find the component quantities of some compound body or amalgam. For instance, you have a mixture of copper and zinc fused together and solidified into one compact body, without a trace of either of the constituents; the weight of each of the respective metals can be submitted to direct calculation, the factors or elements entering into the required formula being merely the specific weight as well of the compound as of each of the ingredients of which it is made up. And so of a mass of quartz and gold; and though little or none of the precious metal may be visible to the eye, the weight of the latter can be arrived at with comparative facility. We are told that Hiero, King of Syracuse, gave to some clever artificer a quantity of gold wherewith to fabricate a crown; but suspecting, when the crown was finished, that the jeweller had purloined a portion of the gold and substituted silver in its stead, he submitted the question to Archimedes to propound. Specific gravities were not then known; but our philosopher, while in his bath, it seems, was cogitating how he might best solve the proposition propounded by the King, when, noticing the difference in weight of his own body when immersed in water to what it was in air, he conceived the happy idea of submitting the crown to a like process of computation, and, after weighing the crown itself in water, and then pure gold and silver, found, by an easy calculation, that, as the King had rightly guessed, the crown was in reality made up of gold and silver instead of gold alone. So glad was our philosopher of the discovery he had made, that he ran through the streets of the city, crying: "Eureka! Eureka!"—"I have found it; I have found it"

Referring, again, to the "*Tableau*," the word *Stereometrical* would seem to imply that it is intended only or altogether for purposes of mensuration. Such, however, is not the case, as it will immediately be evident that it must also be of great utility in acquiring or imparting a knowledge of the nomenclature of solid forms, an acquaintanceship with their varied shapes and figures, which, without such help, would require a previous familiarity with the principles and teachings of drawing and perspective. To the architect, the engineer, and the builder, the models are suggestive of the forms and relative proportions of blocks of buildings, roofs, domes, piers, and quays; cisterns, reservoirs, and cauldrons; vats, casks, and other vessels of capacity; earthworks of all kinds, comprising railroad and other cuttings and embankments; the shaft of the Greek or Roman column; square and waney timber and saw-logs; the camping-tent; the square or splayed opening of a door or window, or niche or loophole in a wall; the quarter of a sphere or spheroid, the half-segment, the vault or arched ceiling of the apsis of a church or hall; the whole sphere or spheroid, the billiard or the cannon ball; or, on a larger scale, the earth, moon, sun, and planets. The models must also prove of great help in teaching perspective drawing and the geometrical projection of solids on a plane; also, their shades and the shadows they project. Again, the art of geometrical development of surfaces is thereby much facilitated. There is also the polar triangle, and other lines necessary for the study of spherical trigonometry.

Whether my attempt to reduce to one simple and uniform system the present multifold and complex rules for finding the contents of solids, or the capacity of space-enclosing areas, shall prove successful, time alone can tell; for, though it has the merit of being new, it also has the disadvantages of novelty, as Mr. Scott-Russell said of the ingenious screw-propeller invented some years ago by Commander Ashe, a most worthy ex-president of this Society. The scientific world, as well as the political, has its conservatives. We have not yet well learnt the advantages of decimal arithmetic; nor has the tiresome computation of pounds, shillings and pence been yet abandoned for the more expeditious dollar, where the mere shifting of a point works wonders. It takes much time to work out such a revolution—a generation, so to say; but that I shall not have to wait so long for an interpreter I confidently hope, judging at least from the many flattering testimonials I have already received in relation to the multiplied advantages of my invention or discovery.

The Council of Public Instruction, at its last general meeting, appointed a committee, composed of the Lord Bishop of Quebec, Bishop Langevin, of Rimouski—himself a thorough master of the art,—and Bishop Larocque, of St. Hyacinthe, to report upon the subject; and who, I take it, after having submitted to them the very favourable opinions of so many of our best mathematicians, and of other competent judges, such as D. Wilkie, R. S. M. Bouchette, the professors of the Laval University, High School, Morrin College, and other educational establishments of Canada and elsewhere, can hardly fail to recommend the introduction of the "*Tableau*" into all the schools of this Dominion.

I have but just received a letter from the Minister of Education of New Brunswick, asking me to send a "*Tableau*," with the view, says he, of introducing it into all the schools of that Province. Not, however, that I in any way intend to confine myself to Canada. On the contrary, I have already patented the "*Tableau*" in the United States, where I hope, of course, to introduce it; and Mr. Vannier, in writing to me from France on the 10th of January last, to advise me of the granting of my letters-patent for that country, adds that MM. Humbert and Noé, the president and secretary of the Society for the Generalization of Education in France, have intimated their intention, at their next general meeting, of having some mark of distinction conferred on me for the benefits which my system is likely to confer on education.

The Honble. Mr. Chauveau, our own worthy Minister of Education, and otherwise well qualified to judge, will make it his duty, so says his letter on the subject, to recommend its adoption in all educational establishments and in every school, so confident he is of its practical utility : " Se fera un devoir d'en recommander l'adoption dans toutes " les maisons d'éducation et dans toutes les écoles, certain " qu'il est de son utilité pratique."

From the Seminary, M. Maingui writes :—" Plus on " étudie, plus on approfondit cette formule du cubage des " corps, plus on est enchanté" (the more one marvels) " de " sa simplicité, de sa clarté, et surtout de sa grande " généralité." Bigelow, M.A., " believing it to be of universal " use, shall heartily lend himself to the introduction of my " system." McQuarrie, B.A., " shall be delighted to see the " old tedious process superseded by a formula so simple and " so exact." D. Wilkie says :—" The rule is precise and " simple, and, being applicable to almost any variety " of solid, will greatly shorten the processes of calculation. " I have," he adds, " proved its accuracy, as applied to " several bodies. The *Tableau*, comprising a great variety " of elementary forms, will serve admirably to educate the " eye, and must greatly facilitate the study of mensuration. " The government would confer a boon on schools of the " middle and higher class by affording access to so " suggestive a collection." Professor Newton, of Yale College, Massachusetts, considers the *Tableau* a very useful arrangement for shewing the variety and extent of the applications of the formula. The Collège L'Assomption " will adopt my system as part of their course of instruction." Rev. T. Boivin, of St. Hyacinthe, says :—" Votre découverte est " précieuse, et je recommande fortement l'adoption de votre " *tableau*, dont je vous prie de me faire connaitre le " coût probable."

There are others who, irrespective of considerations as to the comparative accuracy of the formula, or of its advantages as applied to mere mensuration, have seen how far the models are more suggestive to the pupil and the teacher than the mere representation thereof on the black-board or on paper, and who, in their written opinions, have alluded especially to this feature of the proposed system. Mr. Joly, President of the Quebec branch of the Montreal School of Arts, in a letter to Mr. Weaver on the subject, and after having himself witnessed it on more than one occasion, says, in his expressive style :—" The difference is enormous." The professors at the Normal School are of the same opinion; and others there are who variously estimate the saving in time, by my system, at from twelve to eighteen months.

The prismoidal formula is not new: it has been long known, and sometimes used to compute the ordinary rectangular prismoid, as well as the familiar forms of railroad and canal cuttings and embankments; but no one seems to have conceived the idea of applying it even to the frustum of a cone or pyramid, to which, I must necessarily infer, it was not known to apply,—else, simple as it is, and so much more simple and direct than Legendre's theorem, it must have found its way ere this into treatises on mensuration and the like. Neither has it ever been employed, that I know of, to compute the segment of a sphere or spheroid, nor to many other well-known forms; so that, in this respect, I may lay claim as if to the discovery, and as well for a large number of other solids to which it never was attempted to apply it. And even if the idea of so doing has at any time suggested itself to others, as sometimes hinted at, they do not appear to have put it to the test, or to have arrived at any useful conclusion in relation to it, any more than the first man who, on seeing steam issue under pressure from the nozzle of a tea-kettle, conceived the idea that such an agent could be made to work the wonders that we know of; nor was steam ever made available in practice till Watt invented the steam engine, or electricity till Morse put up a telegraph. Granted, however, that this formula was discovered before my time, and that I have merely disengaged it from the dust of years, or re-discovered it, still is there, perhaps, some little merit to be claimed. Adams and Leverrier both own to the discovery of the planet Neptune; and Leibnitz was not robbed of the honor of his integral and differential calculus, though Newton had by several years preceded him in the field by the discovery of "fluxions."